

# Mode Expansion in the Low-Frequency Range for Propagation Through a Curved Stratified Atmosphere

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This expansion is particularly useful when considering ionospheric propagation at low frequencies. The complex problem dealing with two media, viz., a homogeneous earth and a surrounding stratified atmosphere, leads to intractable expressions. However, as the influence of the earth may be accounted for by an approximate boundary condition at the earth's surface, the problem is then reduced to that of the outer medium only. The coefficients of the mode expansion for this simplified problem will be derived while taking into account the earth's curvature; however, the latter proves to be negligible under very general conditions. The expansion to be derived is wanted in particular when studying the influence of a gradual transition in the electron density with height at the lower edge of the ionosphere.

## 1. Introduction

Maxwell's equations for time-harmonic functions proportional to  $\exp(-i\omega t)$  can be put in the following form (expressed in Gaussian units) provided the dielectric constant  $\epsilon$ , the permeability  $\mu$  and the conductivity  $\sigma$  are independent of time:

$$\left. \begin{aligned} \text{curl } \vec{e} - i\frac{\omega}{c} \mu \vec{h} &= 0; & \text{curl } \vec{h} + i\frac{\omega}{c} \epsilon_c \vec{e} &= 0, \\ \text{div } (\epsilon_c \vec{e}) &= 0; & \text{div } (\mu \vec{h}) &= 0; \end{aligned} \right\}, \quad (1)$$

the complex parameter  $\epsilon_c = \epsilon + i4\pi\Sigma/\omega$  includes both dielectric and conductive properties.

We take  $\mu=1$  and assume a stratified atmosphere of spherical symmetry, all properties of which can be considered as functions of the distance  $r$  to the center of the earth. We accordingly substitute  $\epsilon_c = k^2(r)/k_0^2$ ,  $k_0 = \omega/c$  being the wave number in a vacuum;  $k(r)$  can be interpreted as the local wave number at a height  $r-a$  ( $a$ =earth's radius) above the surface of the earth. In the absence of the earth's magnetic field  $\epsilon_c(r)$  constitutes a scalar. The Maxwell equations (1) can then be satisfied by the field

$$\vec{e} = \frac{i}{k^2(r)} \text{curl curl } \left\{ k(r) \vec{H}r \right\}; \quad \vec{h} = \frac{1}{k_0} \text{curl } \left\{ k(r) \vec{H}r \right\}, \quad (2)$$

provided the Hertzian vector  $\vec{H}r$  has a radial direction throughout space ( $r$  being a vector of length  $r$  in this direction); moreover, the amplitude  $H$  has to satisfy the following scalar wave equation in the outer space  $r > a$ :

$$(\Delta + k_{\text{eff}}^2)H = 0, \quad (3)$$

in which the new parameter  $k_{\text{eff}}^2$  is defined as follows [1]<sup>1</sup>:

$$k_{\text{eff}}^2(r) = k^2(r) - k(r) \frac{d^2}{dr^2} \left\{ \frac{1}{k(r)} \right\}. \quad (4)$$

The solution (2) is of the so-called electric type which includes the case of a vertical electric dipole to be considered hereafter. Let the dipole be placed in a homogeneous slab extending in the region  $a < r < a'$  directly above the earth's surface, the upper boundary  $r=a'$  of which may be identified with the lower edge of the ionosphere in the case of ionospheric propagation.

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<sup>1</sup>Figures in brackets indicate the literature references at the end of this paper.

Assuming a wave number  $k_0$  inside this slab, the primary field of an infinitesimal vertical dipole can be derived from (2) by substituting for  $H$  the function:

$$H_{\text{pr}}(\mathbf{P}) = \frac{Il \exp(i k_0 TP)}{cb TP}, \quad (5)$$

$T$  and  $P$  here represent the positions of transmitter and receiver,  $TP$  their mutual distance,  $b$  the distance from  $T$  to the center of the earth, and  $Il$  the moment of the transmitter. The correctness of (5) is shown by comparing the associated radial Hertzian vector  $H_{\text{pr}} \vec{r}$  with the conventional Hertzian vector for such a dipole. The latter Hertzian vector has a constant direction (parallel to the dipole axis) throughout space, its amplitude being given by

$$\frac{Il \exp(i k_0 TP)}{c TP}.$$

It can be verified that both Hertzian vectors only differ by a gradient, namely that of the quantity  $iIl \exp(i k_0 TP)/(b k_0 c)$ ; however, such a gradient has no effect whatever on the primary field which can therefore also be derived from the radial Hertzian vector  $H_{\text{pr}} \vec{r}$  according to (5).

## 2. Form of the Mode Expansion

The homogeneous wave equation (3) does not hold at the transmitter itself. The latter involves a singularity of the amplitude  $H$  at the point  $T$ . In view of (5) this singularity is the same as that of the function  $Il/(cb TP)$ . We introduce polar coordinates with the origin at the center of the earth and the axis  $\theta=0$  drawn through the transmitter, the position of which therefore is given by  $r=b$ ,  $\theta=0$ . The singularity in question can be accounted for by a right-hand side of (3) which only differs from zero at  $T$ . The complete wave equation in the outer space  $r>a$  thus proves to be:

$$(\Delta + k_{\text{eff}}^2)H = -4 \frac{Il}{cb^3} \delta(b-r) \frac{\delta(\theta)}{\theta}. \quad (6)$$

This equation can be verified by integrating both sides over an infinitesimal sphere around  $T$ , while applying Gauss's integral theorem to the left-hand side, and the following properties of the impulse function to the right-hand side:

$$\int_0^{\theta_0} \frac{\delta(\theta)}{\theta} \sin \theta d\theta = \int_0^{\theta_0} \delta(\theta) d\theta = \frac{1}{2}; \quad \int_{b-\epsilon}^{b+\epsilon} \delta(b-r) dr = 1.$$

We next assume that the influence of the earth may be accounted for by the boundary condition

$$\frac{\partial}{\partial r} (rH) = \gamma rH, \quad (r=a) \quad (7)$$

which is the equivalent of the corresponding condition  $\partial H / \partial z = \gamma H$  for a plane boundary. The quantity  $-i\gamma / \{k_0 \epsilon_c(a)\}$  can be interpreted as a surface admittance (ratio at grazing incidence of the electric field and the magnetic field, both at infinitesimal distances above and below the earth's surface). For  $\epsilon_c(a)=1$  the parameter  $\gamma$  is given, for a vertical dipole, by

$$\gamma = -i \frac{k_0^2}{k_1^2} (k_1^2 - k_0^2)^{1/2},$$

$k_1$  being the complex wave number inside the earth. The special case of a perfectly conducting earth corresponds to  $\gamma=0$ . The value of  $\gamma$  can also be chosen such as to account for a stratification in the ground [2].

The approximative boundary condition (7) has two advantages in the present problem, viz., (a) it reduces the original two-media problem to that of the outer medium only, (b) it guarantees the existence of a complete set of discrete orthogonal modes in the outer space  $a < r < \infty$ . These modes are characterized as solutions of the wave equation (3) that satisfy both the boundary condition (7) at  $r=a$  and a proper radiation condition at  $r=\infty$ .

The general form of the modes is obtained by determining solutions of (3) with the aid of a separation of variables. Owing to the axial symmetry of the present problem around the axis  $\theta=0$  (containing the transmitter) we only have to concern the polar coordinates  $r$  and  $\theta$ . The expansion of the scalar  $H$  in terms of the modes in question then becomes of the type:

$$H = \sum_{l=-\infty}^{\infty} c_l f_{n_l}(r) P_{n_l} \{ \cos (\pi - \theta) \}; \quad (8)$$

the height-gain functions  $f_n(r)$  occurring here are solutions vanishing at infinity of the differential equation

$$\frac{d^2}{dr^2} (rf_n) + \left\{ k_{\text{eff}}^2(r) - \frac{n(n+1)}{r^2} \right\} (rf_n) = 0; \quad (9)$$

finally the discrete eigenvalues  $n_l$  (constituting the order of a Legendre function) are to be determined from the boundary condition

$$\frac{d}{dr} (rf_n) = \gamma rf_n \quad (r=a). \quad (10)$$

### 3. A First Evaluation of the Coefficients of the Mode Expansion

The condition (10) guarantees the required boundary condition (7) for the complete solution (8). All Legendre functions in (8) become infinite along the whole axis  $\theta=0$ ,  $n_l$  being not an integer; however, the proper choice of the coefficients  $c_l$  will reduce this singularity to the required singularity at the transmitter  $T$  only. In view of properties of Legendre functions the singularity of each term of (8) appears from the approximation

$$P_n \{ \cos (\pi - \theta) \} \sim \frac{2}{\pi} \sin (n\pi) \log \frac{\theta}{2} = \frac{2}{\pi} \sin (n\pi) \log \frac{\rho}{2r}, \quad (\theta \rightarrow 0 \text{ or } \rho \rightarrow 0) \quad (11)$$

$\rho$  being the distance from the point  $(r, \theta)$  to the axis  $\theta=0$ . The approximation (11) can be proved by applying Tauber's theorem [3]

$$\lim_{p \rightarrow 0} \left\{ pf(p) \right\} = \lim_{t \rightarrow \infty} \frac{h(t)}{t}$$

to the operational relation  $h(t) \doteq f(p)$  for which [3, p. 243, eq (35)]

$$h(t) = P_n (2e^{-t} - 1) U(t); \quad f(p) = \frac{\Gamma(p) \Gamma(p+1)}{\Gamma(p+n+1) \Gamma(p-n)},$$

while substituting  $t = -2 \log \sin (\theta/2)$ . In its turn the approximation (11) involves the three-dimensional equation:

$$(\Delta + k_{\text{eff}}^2) [f_n(r) P_n \{ \cos (\pi - \theta) \}] = \frac{4}{\pi} \sin (n\pi) \frac{f_n(r)}{r^2} \frac{\delta(\theta)}{\theta}. \quad (12)$$

This equation can be proved along the same lines as (6), but by integrating over an infinitesimal cylinder around  $\theta=0$  instead of an infinitesimal sphere around  $T$ ; the proof is facilitated by replacing  $\delta(\theta)/(r^2\theta)$  by the equivalent quantity  $\delta(\rho)/\rho$ .

A summation of (12) over all eigenvalues, after an initial multiplication by  $c_n$ , yields, in view of (8):

$$(\Delta + k_{eff}^2) H = \frac{4}{\pi} \frac{1}{r^2} \frac{\delta(\theta)}{\theta} \sum_{l=-\infty}^{\infty} c_l \sin(\pi n_l) f_{n_l}(r).$$

The introduction of both negative and positive modes avoids the occurrence of a deviation of  $c_0$  from the general expression for  $c_l$  valid for  $l \neq 0$ . In fact, it is well known from the special case of a perfectly conducting earth and ionosphere that the addition of the  $l$  and  $-l$  mode to a single one involves an exceptional value of  $c_0$  (Neumann factor). Obviously, the right-hand side of the latter relation has to be identical with that of (6), so as to have:

$$\frac{1}{\pi r^2} \sum_{l=-\infty}^{\infty} c_l \sin(\pi n_l) f_{n_l}(r) = -\frac{Il}{cb^3} \delta(b-r).$$

A further multiplication by  $r^2$  yields:

$$\frac{1}{\pi} \sum_{l=-\infty}^{\infty} c_l \sin(\pi n_l) f_{n_l}(r) = -\frac{Il}{cb} \delta(b-r). \quad (13)$$

The right-hand side of (13) can also be expanded in terms of the functions  $f_{n_l}(r)$ . In fact, the complete orthogonality of the latter over the interval  $a < r < \infty$  involves the following expansion for an arbitrary function  $\varphi(r)$ :

$$\varphi(r) = \sum_{l=-\infty}^{\infty} \frac{\int_a^{\infty} \varphi(r) f_{n_l}(r) dr}{\int_a^{\infty} f_{n_l}^2(r) dr} f_{n_l}(r).$$

An application to  $\varphi(r) = \delta(b-r)$  results in:

$$\delta(b-r) = \sum_{l=-\infty}^{\infty} \frac{f_{n_l}(b) f_{n_l}(r)}{\int_a^{\infty} f_{n_l}^2(r) dr}.$$

Hence, by equating the coefficient of  $f_{n_l}(r)$  in both sides of (13), we can evaluate  $c_l$ . The substitution of the value thus found into (8) yields the final form of the mode expansion, viz.,

$$H = -\pi \frac{Il}{cb} \sum_{l=-\infty}^{\infty} \frac{f_{n_l}(b) f_{n_l}(r)}{\int_a^{\infty} f_{n_l}^2(r) dr} \frac{P_{n_l}\{\cos(\pi - \theta)\}}{(\sin \pi n_l)}. \quad (14)$$

#### 4. Another Representation of the Mode Expansion

The coefficients of (14) only depend on the eigen functions  $f_{n_l}$  at the transmitter and receiver, apart from the integral in the denominator. Fortunately, the latter can be reduced as follows to a quantity referring to the earth's surface; hence, the behavior of  $f_{n_l}$  elsewhere needs not to be known explicitly. We apply the operator

$$\frac{\partial}{\partial n} (r f_n) \cdot 1 - r f_n \cdot \frac{\partial}{\partial n}$$

to (9). The resulting expression can be represented in the form:

$$(2n+1) f_n^2 = \frac{d}{dr} \left\{ r f_n \cdot \frac{\partial^2}{\partial r \partial n} (r f_n) - \frac{d}{dr} (r f_n) \cdot \frac{\partial}{\partial n} (r f_n) \right\},$$



in which  $d/dr$  denotes a differentiation with respect to  $r$  at constant  $n$ . An integration of this equation over the interval  $a < r < \infty$ , taking into account the vanishing at infinity of the expression between braces, yields a relation which can be put in the form:

$$(2n+1) \int_a^\infty f_n^2(r) dr = - \left[ r^2 f_n^2 \frac{\partial}{\partial n} \left\{ \frac{d}{dr} \left( \frac{r f_n}{r f_n} \right) \right\} \right]_{r=a}.$$

We bear in mind that  $(d/dr)(rf_n)/(rf_n)$  at  $r=a$  does in general depend on  $n$ , though this quantity assumes the value  $\gamma$  independent of  $n_l$  at the eigenvalues  $n_l$  of  $n$  [see (10)]. The integral in (14) can thus be evaluated; it leads to the following alternative form of the mode expansion for  $H$ :

$$H = \frac{\pi I l}{c a^2 b} \sum_{l=-\infty}^{\infty} \frac{(2n_l+1)}{\left[ \frac{\partial}{\partial n} \left\{ \frac{d}{dr} \left( \frac{r f_n}{r f_n} \right) \right\} \right]_{r=a; n=n_l}} \frac{f_{n_l}(b) f_{n_l}(r)}{f_{n_l}^2(a)} \frac{P_{n_l} \{ \cos(\pi - \theta) \}}{\sin(\pi n_l)}. \quad (15)$$

We now pass to the vertical component  $E_r$  of the electric field, the most important field component for the vertical dipole under consideration. The field representation (2), combined with the wave equation (3), implies the formula:

$$E_r = \frac{i}{k} \left( k_{\text{eff}}^2 + \frac{\partial^2}{\partial r^2} \right) (rH).$$

The operator  $k_{\text{eff}}^2 + \partial^2/\partial r^2$  only affects the height-gain factors  $f_{n_l}(r)$  of (15). The resulting quantity reads, in view of (9),

$$\left( k_{\text{eff}}^2 + \frac{\partial^2}{\partial r^2} \right) (r f_{n_l}) = \frac{n_l(n_l+1)}{r} f_{n_l}.$$

We thus obtain from (15) the following further mode expansion:

$$E_r = \frac{i \pi I l}{k c a^2 b r} \sum_{l=-\infty}^{\infty} \frac{n_l(n_l+1)(2n_l+1)}{\left[ \frac{\partial}{\partial n} \left\{ \frac{d}{dr} \left( \frac{r f_n}{r f_n} \right) \right\} \right]_{r=a; n=n_l}} \frac{f_{n_l}(b) f_{n_l}(r)}{f_{n_l}^2(a)} \frac{P_{n_l} \{ \cos(\pi - \theta) \}}{\sin(\pi n_l)} \quad (16)$$

## 5. Connection of the Mode Amplitude With the Atmospheric Reflection Coefficient

The dependence of the mode coefficients in (16) on the height-gain functions  $f_{n_l}(r)$  can be replaced by that on a properly defined reflection coefficient. In order to discuss this coefficient we assume a homogeneous layer  $a < r < a'$  with wave number  $k = k_0$  above the earth's surface. In the case of ionospheric propagation this layer may be identified with the space between the earth and the ionosphere, its upper boundary  $r = a'$  then representing the lower edge of the ionosphere (at a height  $h = a' - a$  above the earth's surface). In the case of the troposphere the thickness  $h$  might finally approach zero; the wave number in the homogeneous slab then had better be taken equal to  $k'_0 \neq k_0$  in order to account for the small difference of the refractive index at the earth's surface from its vacuum value of unity. We shall restrict ourselves to the ionospheric case; it then is sufficiently accurate to identify the wave number in the inter-space between the earth and the ionosphere with its vacuum value  $k_0$ .

In the homogeneous slab the general solution of (9) is well known, viz., a combination of two spherical Hankel functions:

$$h_n^{(1)}(k_0 r) = \left( \frac{\pi}{2 k_0 r} \right)^{1/2} H_{n+1/2}^{(2)}(k_0 r).$$

The corresponding complete wave functions  $h_n^{(2)}(k_0 r)P_n\{\cos(\pi - \theta)\}$  represent a rising and a descending wave, respectively. A special mode will in general contain both these waves. Hence the height-gain function has the form:

$$f_n(r) = \alpha_n h_n^{(1)}(k_0 r) + \beta_n h_n^{(2)}(k_0 r) \quad (a < r < a'). \quad (17)$$

Near the earth's surface the logarithmic derivative in the denominator of (16) therefore can be represented as follows:

$$\frac{\frac{d}{dr}(rf_n)}{rf_n} = \frac{\alpha_n \frac{d}{dr}\{rh_n^{(1)}(k_0 r)\} + \beta_n \frac{d}{dr}\{rh_n^{(2)}(k_0 r)\}}{r\{\alpha_n h_n^{(1)}(k_0 r) + \beta_n h_n^{(2)}(k_0 r)\}}. \quad (18)$$

This quantity depends, amongst others, on the ratio  $\alpha_n/\beta_n$  which can be connected with a reflection coefficient. In fact, the second term of (17) constitutes the reflected wave that corresponds to the rising wave represented by the first term; hence we can interpret the ratio of both terms at  $r=a'$  as an atmospheric reflection coefficient  $T(n)$ . We then obtain:

$$T(n) = \frac{\beta_n h_n^{(2)}(k_0 a')}{\alpha_n h_n^{(1)}(k_0 a')}, \quad (19)$$

with the aid of which (18) can be transformed as follows at  $r=a$ :

$$\left\{ \frac{\frac{d}{dr}(rf_n)}{rf_n} \right\}_{r=a} = \frac{\frac{d}{dr}\{rh_n^{(1)}(k_0 r)\}_{r=a} + T(n) \frac{h_n^{(1)}(k_0 a')}{h_n^{(2)}(k_0 a')} \cdot \frac{d}{dr}\{rh_n^{(2)}(k_0 r)\}_{r=a}}{a \left\{ h_n^{(1)}(k_0 a) + T(n) \frac{h_n^{(1)}(k_0 a')}{h_n^{(2)}(k_0 a')} h_n^{(2)}(k_0 a) \right\}}. \quad (20)$$

For a stratification degenerating to a homogeneous medium in the domain  $r > a'$  the coefficient  $T(n)$  proves to reduce to the Fresnel reflection coefficient for an elevation angle (compare next section)  $\tau = \arcsin(n/k_0 a)$  if, moreover, the earth's curvature is neglected.

The rigorous expression (20) has to be differentiated with respect to  $n$  in order to obtain the denominator of (16). We shall derive a suitable approximation of this complicated quantity in the next section; it is then convenient to convert (20) into the alternative form:

$$\left\{ \frac{\frac{d}{dr}(rf_n)}{rf_n} \right\}_{r=a} = \left[ \frac{\frac{d}{dr}\{rh_n^{(1)}(k_0 r)\}}{rh_n^{(1)}(k_0 r)} \right]_{r=a} \frac{1 + T(n) \frac{h_n^{(1)}(k_0 a')}{h_n^{(2)}(k_0 a')} \left[ \frac{\frac{d}{dr}\{rh_n^{(2)}(k_0 r)\}}{\frac{d}{dr}\{rh_n^{(1)}(k_0 r)\}} \right]_{r=a}}{1 + T(n) \frac{h_n^{(1)}(k_0 a')}{h_n^{(2)}(k_0 a')} \frac{h_n^{(2)}(k_0 a)}{h_n^{(1)}(k_0 a)}}. \quad (21)$$

## 6. An Approximation of the Mode Expansion (16)

The approximations for the function  $h_n(z)$  have a different analytical form according to the position in the complex plane of the parameter  $z/n$ . We restrict ourselves to those values of  $z/n$  that correspond (for real  $z \sim k_0 a$ ) to eigenvalues  $n_i$  of  $n$  of type I, using Rydbeck's nomenclature [4]. These eigenvalues can be characterized by the inequalities:

$$\operatorname{Re} n < k_0 a - (k_0 a)^{1/3}, \quad \operatorname{Im} n > 0, |n| \gg 1.$$

Such eigenvalues prove to play the dominating role in long-wave propagation. The approximations valid for the corresponding range of  $z/n$  read [5]:

$$h_n^{(1)}(z) \sim \frac{\exp \left\{ (n + \frac{1}{2}) \int_1^{z/n} (1-u^2)^{1/2} \frac{du}{u} \right\}}{(nz)^{1/2} \left( 1 - \frac{z^2}{n^2} \right)^{1/4}}; \quad h_n^{(2)}(z) \sim i \frac{\exp \left\{ -(n + \frac{1}{2}) \int_1^{z/n} (1-u^2)^{1/2} \frac{du}{u} \right\}}{(nz)^{1/2} \left( 1 - \frac{z^2}{n^2} \right)^{1/4}}, \quad (22)$$

in which the square roots are assumed as having a positive real part.

In view of the quantities occurring in (21) we are particularly interested in the two following further approximations which can be derived at once from (22):

$$\frac{h_n^{(1)}(z)}{h_n^{(2)}(z)} \sim -i \exp \left\{ 2(n + \frac{1}{2}) \int_1^{z/n} (1-u^2)^{1/2} \frac{du}{u} \right\}, \quad (23)$$

$$\frac{\frac{d}{dz} \left\{ z h_n^{(1)}(z) \right\}}{z h_n^{(2)}(z)} \sim \frac{1}{2z} + \frac{z}{2(n^2 - z^2)} \pm \frac{(n^2 - z^2)^{1/2}}{z}. \quad (24)$$

We next substitute  $n = k_0 a \sin \tau$  or  $n = k_0 a' \sin \tau'$ . The asymptotic approximations of the complete wave functions  $h_n^{(1)}(k_0 r) P_n\{\cos(\pi - \theta)\}$  then show that  $\tau$  and  $\tau'$  can be interpreted as the (in general complex) angles between the ray trajectories associated with the mode in question, and the verticals at  $r = a$  and  $r = a'$  respectively. Moreover, we substitute for  $z$   $k_0 a$  or  $k_0 a'$ . The corresponding transformations of (23) and (24) lead, amongst others, to the following approximations:

$$\begin{aligned} \frac{h_n^{(1)}(k_0 a')}{h_n^{(2)}(k_0 a')} &\sim i \exp \left[ 2ik_0 \left\{ a' \cos \tau' - a \sin \tau \left( \frac{\pi}{2} - \tau' \right) \right\} \right], \\ \frac{h_n^{(2)}(k_0 a)}{h_n^{(1)}(k_0 a)} &\sim i \exp \left[ -2ik_0 a \left\{ \cos \tau - \sin \tau \left( \frac{\pi}{2} - \tau \right) \right\} \right], \\ \left[ \frac{\frac{d}{dr} \left\{ r h_n^{(1)}(k_0 r) \right\}}{r h_n^{(2)}(k_0 r)} \right]_{r=a} &\sim k_0 \left( \mp i \cos \tau - \frac{\tan^2 \tau}{2k_0 a} \right). \end{aligned}$$

An application of the two latter relations yield:

$$\begin{aligned} \left[ \frac{\frac{d}{dr} \{ r h_n^{(2)}(k_0 r) \}}{\frac{d}{dr} \{ r h_n^{(1)}(k_0 r) \}} \right]_{r=a} &= \left[ \frac{\frac{d}{dr} \{ r h_n^{(2)}(k_0 r) \}}{r h_n^{(2)}(k_0 r)} \right]_{r=a} \times \left[ \frac{\frac{d}{dr} \{ r h_n^{(1)}(k_0 r) \}}{r h_n^{(1)}(k_0 r)} \right]_{r=a}^{-1} \times \frac{h_n^{(2)}(k_0 a)}{h_n^{(1)}(k_0 a)} \\ &= \sim k_0 \left( i \cos \tau - \frac{\tan^2 \tau}{2k_0 a} \right) \times \frac{1}{k_0 \left( -i \cos \tau - \frac{\tan^2 \tau}{2k_0 a} \right)} \times \frac{1}{i \exp \left[ -2ik_0 a \left\{ \cos \tau - \sin \tau \left( \frac{\pi}{2} - \tau \right) \right\} \right]}. \end{aligned}$$

The final substitutions of all these approximations into (21) yields:

$$\left\{ \frac{\frac{d}{dr} (r f_n)}{r f_n} \right\}_{\tau=a} \sim - \left( ik_0 \cos \tau + \frac{\tan^2 \tau}{2a} \right) \frac{1 - T(n) e^{2ik_0 h \cos \tau} \frac{1 + i \tan^2 \tau / (2k_0 a \cos \tau)}{1 - i \tan^2 \tau / (2k_0 a \cos \tau)}}{1 + T(n) e^{2ik_0 h \cos \tau}}, \quad (25)$$

provided we neglect the small difference between  $\tau$  and  $\tau'$  (this difference vanishes altogether in the flat-earth approximation).

In (16) we need the derivative of (25) with respect to  $n$ . According to the relation  $n = k_0 a \sin \tau$  we have

$$\frac{\partial}{\partial n} = \frac{1}{k_0 a \cos \tau} \frac{\partial}{\partial \tau}.$$

A corresponding evaluation of the  $n$ -derivative of (25) results in:

$$\frac{\partial}{\partial n} \left\{ \frac{\frac{d}{dr} (rf_n)}{rf_n} \right\}_{\tau=a} = \sim \frac{1}{a \cos \tau} \left\{ i \sin \tau - \frac{\sin \tau}{k_0 a \cos^3 \tau} - \left( i \cos \tau + \frac{\tan^2 \tau}{2k_0 a} \right) \frac{\partial}{\partial \tau} \right\} \frac{1 - T(n) e^{2ik_0 h \cos \tau} \frac{1 + i \tan^2 \tau / (2k_0 a \cos \tau)}{1 - i \tan^2 \tau / (2k_0 a \cos \tau)}}{1 + T(n) e^{2ik_0 h \cos \tau}}. \quad (26)$$

Moreover, we shall apply the well-known asymptotic approximation for Legendre functions having a large complex order  $n$  with positive imaginary part, viz.,

$$\frac{P_n \{ \cos (\pi - \theta) \}}{\sin (n\pi)} \sim - \left( \frac{2i}{\pi n \sin \theta} \right)^{1/2} e^{i(n+1/2)\theta}. \quad (27)$$

A final substitution of both (25) and (27) into (16), while approximating  $n_l + 1$  and  $n_l + 1/2$  by  $n_l = k_0 a \sin \tau_l$ , yields the following expansion in the simplest case of a transmitter and receiver on the ground ( $r = b = a$ ):

$$E_r \sim -2 \frac{k_0^{3/2} I l}{c} \left( \frac{2i\pi}{a \sin \theta} \right)^{1/2} e^{i\theta/2} \sum_{l=-\infty}^{\infty} \frac{\sin^{5/2} \tau_l e^{ik_0 a \theta \sin \tau_l}}{\left\{ \tan \tau + \frac{i \tan \tau}{k_0 a \cos^3 \tau} - \left( 1 - \frac{i \sin^2 \tau}{2k_0 a \cos^3 \tau} \right) \frac{\partial}{\partial \tau} \right\} \left\{ \frac{1 - T(n) e^{2ik_0 h \cos \tau} \frac{1 + i \sin^2 \tau / (2k_0 a \cos^3 \tau)}{1 - i \sin^2 \tau / (2k_0 a \cos^3 \tau)}}{1 + T(n) e^{2ik_0 h \cos \tau}} \right\}}_{\tau=\tau_l} \quad (28)$$

## 7. Flat Earth Approximation of the Mode Expansion

The following simplification results from (28) for  $a = \infty$ , remembering the relation  $\theta = d/a$  ( $d$  = distance from transmitter to receiver):

$$E_r \sim -2 \frac{k_0^{3/2} I l}{c} \left( \frac{2\pi i}{d} \right)^{1/2} \sum_{l=-\infty}^{\infty} \frac{\sin^{5/2} \tau_l e^{ik_0 d \sin \tau_l}}{\left[ \left( \tan \tau - \frac{\partial}{\partial \tau} \right) \left\{ \frac{1 - T(n) e^{2ik_0 h \cos \tau}}{1 + T(n) e^{2ik_0 h \cos \tau}} \right\} \right]_{\tau=\tau_l}}. \quad (29)$$

This expression can also be derived straightforwardly by applying the flat earth conditions right from the beginning, while replacing the boundary condition (7) at the earth's surface by

$$\frac{\partial \Pi}{\partial z} = \gamma \Pi \quad (z=0). \quad (30)$$

The rigorous expansion for the amplitude of the vertically directed Hertzian vector  $\Pi$  [to be substituted in (2) instead of  $H\vec{r}$ ] then arrived at reads:

$$\Pi = \frac{2\pi i}{c} I l \sum_{l=-\infty}^{\infty} \frac{\lambda_l}{\left[ \frac{\partial}{\partial \lambda} \left\{ \frac{df_\lambda/dz}{f_\lambda} \right\} \right]_{z=0; \lambda=\lambda_l}} \frac{f_{\lambda_l}(h_0) f_{\lambda_l}(z)}{f_{\lambda_l}^2(0)} H_0^{(1)}(\lambda_l d). \quad (31)$$

In this series, replacing (15),  $f_\lambda(z)$  denotes a height-gain function solving the differential equation

$$\frac{d^2 f_\lambda}{dz^2} + \{k_{\text{eff}}^2(z) - \lambda^2\} f_\lambda = 0,$$

and satisfying the radiation condition at infinity ( $z = \infty$ ). The other boundary condition (30) involves the relation

$$f'_\lambda(0) = \gamma f_\lambda(0) \quad (z=0) \quad (32)$$

at the earth's surface [replacing (7)]; this condition can only be satisfied if  $\lambda$  equals one of the eigenvalues  $\lambda_l$ ;  $h_0$  represents the height of the transmitter above the earth. The expansion for the vertical component of the electric field [replacing (16)], to be derived from (31), becomes:

$$e_z = -\frac{2\pi}{k_0 c} H \sum_{l=-\infty}^{\infty} \frac{\lambda_l^3}{\left[ \frac{\partial}{\partial \lambda} \left\{ \frac{df_\lambda/dz}{f_\lambda} \right\} \right]_{z=0; \lambda=\lambda_l}} \frac{f_{\lambda_l}(h_0) f_{\lambda_l}(z)}{f_{\lambda_l}^2(0)} H_0^{(1)}(\lambda_l d). \quad (33)$$

The wave functions in the homogeneous slab  $0 < z < h$  above the earth here reduce to simple exponentials so that (17) is to be replaced by:

$$f_\lambda(z) = \alpha_\lambda \exp\{-(\lambda^2 - k_0^2)^{1/2} z\} + \beta_\lambda \exp\{(\lambda^2 - k_0^2)^{1/2} z\}, \quad (34)$$

which enables an easy evaluation of the denominator in (29). The final series representing  $e_z$  for a transmitter and receiver on the ground ( $h_0 = z = 0$ ) then proves to be, substituting more-over  $\lambda_l = k_0 \sin \tau_l$ ,

$$e_z = -\frac{2\pi i k_0^2}{c} H \sum_{l=-\infty}^{\infty} \frac{\sin^3 \tau_l}{\left[ \left( \tan \tau - \frac{\partial}{\partial \tau} \right) \left\{ \frac{1 - T(\tau) e^{2ik_0 h \cos \tau}}{1 + T(\tau) e^{2ik_0 h \cos \tau}} \right\} \right]_{\tau=\tau_l}} H_0^{(1)}(k_0 d \sin \tau_l). \quad (35)$$

This rigorous series leads to (29) if the Hankel function is replaced by its asymptotic approximation. A comparison of (29) and (28) shows under what circumstances the earth's curvature becomes negligible.

An evaluation of the derivative in the denominator of (35) leads to the alternative form:

$$e_z = -\frac{2\pi i k_0^2}{c} H \sum_{l=-\infty}^{\infty} \frac{\sin^3 \tau_l}{\left[ \tan \tau \frac{1 - T e^{2ik_0 h \cos \tau}}{1 + T e^{2ik_0 h \cos \tau}} - 2 \frac{H_0^{(1)}(k_0 d \sin \tau_l) e^{2ik_0 h \cos \tau} (2ik_0 h \sin \tau T - \partial T / \partial \tau)}{(1 + T e^{2ik_0 h \cos \tau})^2} \right]_{\tau=\tau_l}} \quad (36)$$

The denominator of (36) can further be worked out by considering the equation determining the modes. The latter is arrived at as follows. A substitution of (34) while evaluating the boundary condition (32) leads to the relation

$$\frac{(\beta/\alpha) - 1}{(\beta/\alpha) + 1} (\lambda^2 - k_0^2)^{1/2} = \gamma \text{ for } \lambda = \lambda_l.$$

On the other hand, the definition of the atmospheric reflection coefficient for the flat case involves the following analogy of (19):

$$T(\lambda) = \frac{\beta}{\alpha} e^{2h(\lambda^2 - k_0^2)^{1/2}}.$$

An elimination of  $\beta/\alpha$  from the two latter relations yields an equation which can be put in the following form [replacing  $\lambda$  by  $k_0 \sin \tau$  and  $(\lambda^2 - k_0^2)^{1/2}$  by  $-ik_0 \cos \tau$ ]:

$$\frac{ik_0 \cos \tau_l + \gamma}{ik_0 \cos \tau_l - \gamma} \cdot T(\tau_l) e^{2ik_0 h \cos \tau_l} = 1.$$

The first quotient here proves to be nothing else but the approximative plane-wave reflection coefficient at the earth's surface,  $R(\tau_l)$  say, that results from the boundary condition (32). Hence the mode equation reads:

$$R(\tau_l) T(\tau_l) e^{2ik_0 h \cos \tau_l} = 1. \quad (37)$$

This simple equation is also obtained at once from the following resonance condition. We consider a plane rising wave in the homogeneous slab  $0 < z < h$ , and derive the other plane rising wave that results from it after a reflection against the lower edge of the stratification and a reflection thereafter against the earth surface. The relation (37) then expresses the condition that the latter rising wave should be identical to the original one.

The exponential  $\exp(2ik_0 h \cos \tau_l)$  in (36) can now be replaced throughout by  $\{R(\tau_l) T(\tau_l)\}^{-1}$ . This yields the following expansion replacing (36):

$$e_z = \frac{2\pi ik_0^2}{c} Il \sum_{l=-\infty}^{\infty} \frac{\{1 + R(\tau_l)\}^2 \sin^3 \tau_l}{\{(1 - R^2) \tan \tau + 2R(2ik_0 h \sin \tau - \partial \log T / \partial \tau)\}_{\tau=\tau_l}} H_0^{(1)}(k_0 d \sin \tau_l). \quad (38)$$

For completeness sake we also mention the corresponding asymptotic approximation based on large values of  $k_0 d$ , viz.,

$$e_z \sim \frac{(\pi l)^{1/2} (2k_0)^{3/2}}{c} \frac{Il}{d^{1/2}} \sum_{l=-\infty}^{\infty} \frac{\{1 + R(\tau_l)\}^2 \sin^{3/2} \tau_l \cos \tau_l}{\left\{1 - R^2 + 2R \cos \tau \left(2ik_0 h - \frac{\partial \log T / \partial \tau}{\sin \tau}\right)\right\}_{\tau=\tau_l}} e^{ik_0 d \sin \tau_l}.$$

## 8. Special Case of a Flat Stratification Reducing to a Homogeneous Atmosphere

A further reduction of (38) is only possible by introducing a special model for the stratification in the space  $z > h$ . Let us consider the example of a homogeneous atmosphere, sharply bounded at  $z = h$ . The other boundary  $z = 0$  of the interspace  $0 < z < h$  also being sharp, the situation becomes almost identical to that of a wave guide limited by two infinite plane parallel plates with a separation  $h$ . However, we have to bear in mind the approximative boundary condition applied at  $z = 0$  so that this wall behaves differently from the other wall at  $z = h$ . Let us mark the refractive index of the atmosphere by  $n_i$ ; the reflection coefficient  $T$  then reduces to an expression of the Fresnel type, viz.,

$$T(\tau) = \frac{n_i^2 \cos \tau - (n_i^2 - \sin^2 \tau)^{1/2}}{n_i^2 \cos \tau + (n_i^2 - \sin^2 \tau)^{1/2}}.$$

This formula involves the relation

$$\frac{\partial \log T}{\partial \tau} = \frac{2 \tan \tau (1 + T)}{(1 - T)(\sin^2 \tau - n_i^2 \cos^2 \tau)},$$

with the aid of which (38) can further be transformed [6] into:

$$e_z = \frac{2\pi ik_0^2}{c} Il \sum_{l=-\infty}^{\infty} \left[ \frac{(1 + R)^2 \sin^2 \tau \cos \tau}{1 - R^2 + 4R \left\{ ik_0 h \cos \tau - \frac{1 + T}{(1 - T)(\sin^2 \tau - n_i^2 \cos^2 \tau)} \right\}} \right]_{\tau=\tau_l} H_0^{(1)}(k_0 d \sin \tau_l). \quad (40)$$

Owing to the approximative boundary condition at  $z=0$  we here obtain a completely discrete sum. The rigorous boundary condition at  $z=0$ , on the contrary, would yield also an integral contribution which can be ascribed to a continuous part of the mode spectrum. However, the latter vanishes in the case of a perfectly conducting wall at  $z=0$ .

The result (40) is also rigorous for a wave guide limited by two perfectly conducting walls at  $z=0$  and  $z=h$ ; we then have  $T=R=1$ . In this case we derive from (40) the following series for the vertical field component at  $z=0$  that is due to a dipole also situated at  $z=0$ :

$$e_z = \frac{2\pi k_0}{ch} Il \sum_{l=-\infty}^{\infty} \sin^2 \tau_l H_0^{(1)}(k_0 d \sin \tau_l).$$

The corresponding asymptotic approximation becomes:

$$e_z \sim 2^{3/2} \left( \frac{\pi k_0}{id} \right)^{1/2} \frac{Il}{ch} \sum_{l=-\infty}^{\infty} \sin^{3/2} \tau_l H_0^{(1)}(k_0 d \sin \tau_l);$$

moreover, we may substitute  $\tau_l = \arccos(\pi l / k_0 h)$ .

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## 9. References and Notes

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